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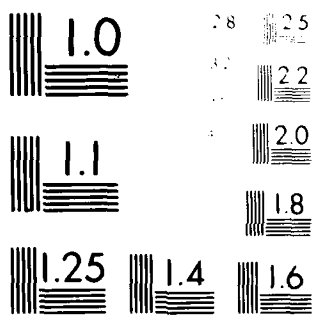
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ANNUAL TECHNICAL REPORT

NONLINEAR WAVE PROPAGATION

AFOSR Grant [redacted] AFOSR-78-3674

by

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and

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The basic direction underlying this work is the continuing study of certain fundamental features associated with the nonlinear wave propagation which are motivated and arise in physical problems. The usefulness of the work is attested to by the varied applications, and wide areas of interest in physics, engineering and mathematics. The work accomplished relates to fluid mechanics, nonlinear optics, multidimensional solitons, Painlevé equations, long time asymptotic solutions, new linearizations of the KdV equation, Riemann-Hilbert problems, etc.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

A. Abstract

The basic direction underlying this work is the continuing study of certain fundamental features associated with the nonlinear wave propagation which are motivated and arise in physical problems. The usefulness of the work is attested to by the varied applications, and wide areas of interest in physics, engineering and mathematics. The work accomplished relates to fluid mechanics, nonlinear optics, multidimensional solitons, Painlevé equations, long time asymptotic solutions, new linearizations of the KdV equation, Riemann-Hilbert problems etc.

(1) Research Objectives and Accomplishments

The continuing theme of the work performed under this grant has been the study of nonlinear wave propagation associated with physically significant systems. The work has important applications in fluid dynamics (e.g. long waves in stratified fluids), nonlinear optics (e.g. self-induced tranparency, and self-focussing of light), and mathematical physics as well as important consequences in mathematics. Individuals working with me and hence partially associated with this grant include: Dr. [illegible]ssios Fokas, Assistant Professor of Mathematics and Computer Science, Dr. Akira Nakamura, post doctoral fellow in Mathematics and Computer Science, Mr. Thiab Taha and Mr. Paulo Santini, graduate students in Mathematics and Computer Science. Attached please find the technical section of our recent proposal to A.F.O.S.R. In this proposal many of the main directions and results are outlined. In addition we have listed all our publications and preprints carried out and conceived during the period of this grant.

Finally we point out an entirely new and important area of study which was developed during the past year. Namely we have employed a novel linear integral equation which, in principal allows one to capture a far larger class of solutions to the Korteweg-deVries equation (KdV) than does the well known Gel'fand-Levitan-Marchenko equation. In our recent paper attached we (a) give a direct proof of the above facts; (b) discuss how the Gel'fand-Levitan-Marchenko equation may be recovered as a special case, (c) characterize a three parameter family of solutions to the self-similar O.D.E. associated with KdV and which may be directly related to the second Painlevé equation. (We note that the Gel'fand-Levitan'Marchenko equation only characterizes a one parameter family of such solutions). In order to carry out (c) we had to investigate a concrete singular integral equation in which the contour L consists of 5 rays all passing thru the origin. The analysis requires the full power (and some extensions) of the classical theory of singular integral equations.

It should be remarked that (i) the integral equation applies to potentials of the Schrödinger equation, even without the application to KdV or P_{II}; (ii) the motivation for developing such an integral equation originates from the concept of summing perturbation series. (iii) Recently Flaschka and Newell considered P_{II} via monodromy theory. In their work they derive a formal system of linear singular integral equations for the general solution of P_{II}. However the highly nontrivial question of existence of solutions was left open. How their work and ours relate is a question for future research. (iv) The linear version of KdV: $u_t+u_{xxx} = 0$ is solved in full generality as a special case.

Some future directions are: (a) Investigation of the full generality of the solutions of KdV via this new formulation. (b) Development of similar types of integral equations for other nonlinear evolution equations, as well as ones which relate to natural "equilibrium" states for KdV, other than the zero (or vacuum) state

The other areas of interest are continuing. They include:

- Investigation of a class of physically significant nonlinear singular integro-differential equations, and associated novel scattering problems.
- Transverse Instability of One-Dimensional Transparent Optical Pulses in Resonant Media.
- Perturbations of Solitons and Solitary Waves.
- Focussing Phenomena in Nonlinear Wave Propagation.
- Two-dimensional Lumps and Multidimensional I.S.T.
- A Connection between Nonlinear Evolution Equations and Nonlinear O.D.E.'s of Painlevé Type.
- Discrete I.S.T. and Numerical Applications.
- Long Time Asymptotic Solutions.
- Applications of Hirota's Bilinear Theory.

2. Recent Renewal Proposal

Renewal of Research Funding

Submitted to

Air Force Office of Scientific Research

Nonlinear Wave Propagation

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and

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0. Forward

The main purpose for the continuation of this research funding is to support the work presently being carried out by Mark Ablowitz, and his associates, in the Mathematics and Computer Science Department at Clarkson College of Technology. The principal investigator has been working in the general area of nonlinear wave propagation for almost ten years. The scope of the work is broad, although it has as its principal focus the understudying of nonlinear phenomena connected with the wave propagation which arise in physical problems. In recent years significant breakthroughs have been made and this area of research is of current interest to mathematicians, physicists, and engineers alike. During the past year, the active research funds allowed us to support Dr. J. Satsuma as a Postdoctoral Research Associate at Clarkson, while he was on leave from Kyoto University. Dr. Satsuma is a well known authority in this field of research and he has been a valuable asset to our research program. Dr. Satsuma returned to Kyoto on September 30, 1980; nevertheless we have continued to collaborate on problems of joint interest. Subsequently Dr. Akira Nakamura, also from Kyoto has come to Clarkson as a Postdoctoral Associate on this grant. Dr. Nakamura has written a number of significant papers in this area and we expect collaboration between us to be very fruitful.

The proposal is divided as follows. In the first section an abstract of the research is given. In the second section we give a report of current and proposed research. The third section gives references; the fourth section contains curriculum vitaes of the principal investigator and Dr. Nakamura, and the fifth section contains a proposed budget for two years.

1. Abstract

During the past two decades significant advances in the study of nonlinear wave phenomena have occurred. These advances have allowed researchers to begin to understand some of the fundamental building blocks associated with nonlinear waves as well as being able to obtain solutions to a number of nonlinear evolution equations. It is important to recognize that these studies are generic in nature and apply to numerous physical problems such as propagation of long waves in stratified fluids, self-focussing in nonlinear optics, self-induced transparency, water waves, plasma physics, etc.

In the period of time mentioned above, both approximate and exact methods of solution to problems of physical significance have emerged. Especially significant amongst the exact methods of analysis is what I shall refer to as the Inverse Scattering Transform and the associated concept of the soliton. This method has found applications to physics, engineering and mathematics alike. The results already obtained, and the wide ranging interest in these problems have motivated our work. In this proposal we discuss some of the research problems which we are actively pursuing.

2. Current and Proposed Research

Since this reserch began to be supported by the Air Force Office of Scientific Research, we have actively studied a number of problems in nonlinear wave theory. In what follows we shall list some of the areas which we have studied along with the principal results and future directions.

(a) A Class of Physically Significant Singular Nonlinear Integro-Differential Equations.

Very recently we have become interested in a class of nonlinear singular integro-differential equations. The particular physical application of this problem is to long internal gravity waves in a stratified fluid. However both the way in which it arises, and the relevant mathematics strongly suggest that many other applications will be found as well. In fact private communications have indicated that there are applications to plasma physics. The specific equation we have considered is:

$$u_t+2uu_x+T(u_{xx})+\frac{1}{\delta}u_x = 0 \qquad (1)$$

where $$T(u) = ⨍_{-\infty}^{\infty}\left(-\frac{1}{2\delta}\right)\coth\left(\frac{x-\xi}{2\delta}\right)u(\xi)\,d\xi.$$

$⨍_{-\infty}^{\infty}$ represents the principal value integral and δ is a parameter. References [1,2] discuss the derivation of (2) in the context of internal waves. As $\delta\to 0$ we have the KdV equation

$$u_t+2uu_x+\frac{\delta}{3}u_{xxx} = 0, \qquad (2)$$

wheras if $\delta\to\infty$ we have the so-called Benjamin-Ono equation

$$u_t+2uu_x+H(u_{xx}) = 0, \tag{3}$$

where $H(u) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi)}{\xi-x}d\xi$ is the Hilbert transform of u.

Thus equation (1) contains as limiting forms both the KdV and Benjamin-Ono equations. The fact that (1) has multi-soliton solutions ([3], [4]) suggested to us that indeed (1) may be solvable by the Inverse Scattering Transform (I.S.T.). In our recent work [5], [6] we have found a Bäcklund Transformation, a generalized Miura Transformation, soliton and rational solutions, interesting dynamical systems and a <u>new type of scattering problem</u>. This scattering problem is given by the equation

$$i\psi_x^+ + (u-\lambda)\psi^+ = \mu\psi^- \tag{4}$$

where u satisfies equation (1), and $\psi^\pm$ are the boundary values of a function analytic in the strips $o<\mathrm{Im}x<2\delta$ for ψ^+, $-2\delta<\mathrm{Im}x<o$ for ψ^-, and periodically extended. Specifically, equation (4) is a differential Riemann-Hilbert problem. When λ,μ are given by

$$\lambda = -k\coth 2k\delta, \quad \mu = k\,\mathrm{cosech}\, 2k\delta,$$

and $\psi^-(x) = \psi^+(x+2i\delta)$ (by periodicity) we find that in the limit $\delta\to o$ we have the Schrödinger scattering problem

$$\psi_{xx}+(k^2+u/\delta)\psi = 0 \tag{5}$$

which is the linear scattering problem associated with the KdV equation (2). At the present time we are studying the inverse scattering associated with (4). In our most recent work [7] we have found Fredholm integral equations for the Jost functions associated with (4). This is unlike previous local scattering problems where the Jost functions satisfy Volterra type integral equations. We note that now the scattering problem (4) and associated nonlinear evolution equation (1) are not local. In those cases where the Fredholm equations have no nontrivial homogeneous solutions, we have been able to do the inverse scattering and hence the corresponding initial value problem associated with (1). This requires both δ and $\max|u(x)t=0)|$ to be small enough (i.e., satisfy certain inequalities - in some sense this is near the KdV equation). When $\delta\to\infty$ (the Benjamin-Ono limit) we have found homogeneous solutions to the Fredholm integral equation. We have not yet carried out the complete inverse scattering analysis when such homogeneous solutions exist. This will be one important aspect of our future work.

Finally, it should be pointed out that we feel that there are other significant nonlinear singular integro differential evolution equations which should fall into the category of solvable by I.S.T. and processing solitons, we shall also investigate such possibilities in the near future.

(b) Transverse Instability of One Dimensional Transparent Optical Pulses in Resonant Media.

It is well known that ultrashort optical pulses may propagate coherently, without attenuation in certain resonant media [8,9]. This phenomena is commonly referred to as Self-Induced Transparency (S.I.T.) and has been intensively studied experimentally, numerically, and analytically by numerous researchers, motivated at least in part, by significant potential applications. From a mathematical point of view the one dimensional equations of S.I.T. are very special. Namely, it has been shown that these equations can be fully integrated by the use of the Inverse Scattering Transform [10,11]. Specifically, the above analysis has shown that arbitrary initial values break up into a sequence of coherent pulses, which do not decay as they propagate, plus radiation which rapidly attentuates. These coherent pulses are referred to as solitons.

There are various types of solitons [8,9]; e.g. "2π pulses" ("hyperbolic secant pulses"), "$o\pi$ pulses" ("breathers") etc. In our paper, "Transverse Instability of One-Dimensional Transparent Optical Pulses in Resonant Media", [12] we have shown analytically, that the 2π pulse is, in fact, unstable to certain transverse variations (i.e. multidimensional perturbations). These results are consistent with numerical and experimental studies on the transverse effects in S.I.T. [13-14]. The latter work has shown that transverse variations can lead to frequency-amplitude modulations, and in some cases self-focussing filaments. Similarly in [15] we have recently been able to show that the breather solution ($o\pi$ pulse) is also unstable to long transverse perturbations. Mathematically speaking, this work was difficult because the earlier analysis had to be much further developed. We point out that this analytical stability calculation is on a mode which is much more complicated than a permanent travelling wave (i.e. a simple soliton 2π pulse). In the future we wish to examine the stability of a double pole solution (i.e. a limiting form of a breather solution just before it breaks up into a two soliton state) as well as attempting to more fully understand both the properties of the two dimensional model, as well as looking for new multidimension soliton like solitons (see also (e)).

(c) Perturbations of Solitons and Solitary Waves.

The above work on transverse stability of solitons in S.I.T. led us naturally to the problem of adding general weak perturbations to equations which admit solitons or solitary waves as special solutions (both in one and more than one dimension). Some of the mathematical machinery was already in place due to the work done in part (b) described above. We have found [16] that, generally speaking, such perturbation problems can be successfully handled by more or less well known perturbation methods. We have compared our results to some of those in the literature which employ the Inverse Scattering Transform (see for example [17-19]. One advantage to our technique is that it also applies to problems for which I.S.T. does not apply.

Our analysis shows in some detail that there is quite different phenomena occurring in different regions of space. Namely near the peak of the soliton we have adiabatic motion of the soliton (or solitary wave). Away from the soliton a linear W.K.B. theory applies. The results are asymptotically matched in order to obtain a uniformly valid theory. To our knowledge this theory is the first such uniformly valid calculation of a perturbation of a soliton or solitary wave. Previous theories were valid in limited regions of space only.

By examining other equations admitting solitary wave solutions (i.e. ones which are not solvable by I.S.T.) we believe that we have discovered a new class of equations which have focussing sngularities, (namely, equations which have certain solutions which are "nice" initially, but blow up in a finite time).

For example, we have discovered evidence that strongly indicates that the following equation is in this class:

$$u_t+u^p u_x+u_{xxx} = 0 \tag{6}$$

for $p \geqslant 4$. We hope to continue to investigate such questions in the near future. These questions are of both mathematical and physical interest. For example, such a question arises in the propagation of water waves ((d) below).

(d) Focussing and instability associated with the propagation of water waves.

As mentioned, one such physical problem where focussing occurs is that of water waves. We have considered the evolution of gravity-capillary waves on a free surface of a layer of fluid with constant depth. Beginning with the standard equations of water waves, one can develop the evolution equation of a modulated weakly nonlinear periodic wave (with fixed central wave number k) travelling in the x-direction whose amplitude and phase vary slowly in both the x and y directions. This problem has been considered by Benney and Roskes [20] and Davey and Stewartson [21] without surface tension and by Djordjevic and Redekopp [22] who included this effect. For sufficiently long times, the complex wave amplitude $A(x,y,t)$ and mean velocity potential $\Phi(x,y,t)$ satisfy

$$iA_t + \sigma_1 A_{xx} + A_{yy} = \sigma_2 |A|^2 A + \Phi_x A$$

$$\alpha\Phi_{xx} + \Phi_{yy} = \beta(|A|^2)_x \qquad (7)$$

Here, $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1, \alpha, \beta$ are real constants which depend on a dimensionless wavenumber and dimensionless surface tension coefficient. Ablowitz and Segur [23] have shown that there exists a whole parameter regime where these equations have solutions which evolve from "nice" initial values into a state where the amplitude becomes infinite in finite time! In this regime the fully nonlinear water wave equations must be analyzed. Moreover, in [23] it was demonstrated that all one dimensional soliton solutions to (7) are unstable with respect to long transverse variations. These results indicate the need for understanding the true multidimensional character of the equation. It also should be noted that the work in [23] has motivated recent experimental work. In the future we hope to consider how these focussing solutions can be understood in the context of the fully nonlinear water wave equations, as well as studying potentially solvable multidimensional cases of (7) (see also (e) below).

(e) Two-dimensional Lumps in Nonlinear Dispersive Systems

Suitable (long wave) limits of the above equation of governing water waves (7) [21-22] reduce to

$$iA_t - \sigma_1 A_{xx} + A_{yy} = \sigma_2 A|A|^2 + 2\sigma_1\sigma_2\phi_x A$$

$$\sigma_1\phi_{xx} + \phi_{yy} = -(|A|^2)_x \tag{8}$$

σ_1, $\sigma_2 = \pm 1$, or for very long waves (comparable to the weak nonlinearities):

$$(u_t + 6uu_x + u_{xxx})_x + \sigma_3 u_{yy} = 0, \tag{9}$$

$\sigma_3 = \pm 1$. Both of these equations belong to the class of nonlinear evolution equations where I.S.T. is applicable [24,25]. Indeed, recent work on the complete integration of these and also the three wave interaction equations has been undertaken very recently [see for example 26,27,28]. Alternatively, via Hirota's method, N plane wave soliton solutions can be directly constructed [see for example 29]. These latter solutions are quasi-one dimensional; they describe the multiple collision of N solitons each of which may propagate in different directions, but which do not decay at infinity.

It is of interest to find essentially two-dimensional solitons which would be analogues of those in one dimension. By taking limits of the N-plane wave soliton solutions described above and choosing certain parameters appropriately, we have discovered permanent two-dimensional nonsingular lump type solutions decaying in all directions to equation (9) [30, see also 31]. Continuing in that direction we have investigated analogous solutions associated with equation (8). We have shown that eq. (8) possesses a lump solution of envelope hole type. Moreover, we have constructed solutions describing multiple collisions of lumps to both eqs. (8), (9) [32]. It is of interest to see how these solutions can be fit into the I.S.T. picture and also to investigate other equations which may give rise to such multidimensional lump type solutions (e.g. S.I.T., the self dual Yang Mills equations etc.). We hope to investigate such questions in the future as well as to study the generality of the initial value solutions of such equations obtained by I.S.T.

(f) A Connection Between Nonlinear Evolution Equations and Certain Nonlinear O.D.E.'s of Painlevé type

The development of the inverse scattering transform (I.S.T.) has shown that certain nonlinear evolution equations possess a number of remarkable properties, including the existence of solitons, an infinite set of conservation laws, an explicit set of action angle variables, etc. We have noted in [33] that there is a connection between these nonlinear partial differential equations (PDE's) solvable by I.S.T. and nonlinear ordinary differential equations (ODE's) without movable critical points. (Some definitions: a <u>critical point</u> is a branch point or an essential singularity in the solution of the ODE. It is <u>movable</u> if its location in the complex plane depends on the constants of integration of the ODE. A family of solutions of the ODE without movable critical points has the <u>P-property</u>; here P stands for Painlevé.) In [34-36] we have announced and developed a number of results which indicate that this connection to ODE's of P-type is yet another remarkable property of these special nonlinear PDE's.

We have conjectured that:

> Every nonlinear ODE obtained by a similarity reduction of a nonlinear PDE of I.S.T. class is, perhaps after a transformation of variables, of P-type.

Here we refer to a nonlinear PDE as being in the I.S.T. class if nontrivial solutions of the PDE can be found by solving a linear integral equation of the Gel-fand-Levitan-Marchenko form. No general proof of this conjecture is available yet, but we have proven a more restricted result in this direction. It is known that under scaling transformations certain nonlinear PDE's of I.S.T. class reduce to ODE's. Moreover, the solutions of these ODE's may be obtained by solving linear integral equations. We have shown that every such family of solutions has the P-property.

We note that the conjecture in its strongest form relates to ODE's obtained from equations solved directly by I.S.T. There are many examples of equations solved only indirectly by I.S.T.; the sine-Gordon equation is one of the best known examples. An ODE obtained from an equation solved indirectly by I.S.T. need not be of P-type, but it may be related through a transformation to an ODE that is.

One consequence of this conjecture is an explicit test of whether or not a given PDE may be of I.S.T. class; namely, reduce it to an ODE, and determine whether the ODE is of P-type. To this end, we identify certain necessary conditions that an ODE must satisfy to be of P-type and describe an explicit algorithm to determine whether an ODE meets these necessary conditions.

Finally, we have exploited this connection in order to develop both solutions and asymptotic connection formulae to some of the classical transcendents of Painleve [37] as well as others.

In the future we shall consider the following problems: (1) the complete connection formulae (i.e. the global connection of asymptotic states) for the interesting Painlevé equations associated with linear Gel'fand-Levitan-Marchenko equations; (2) To prove that the O.D.E.'s which we have derived, in fact satisfy the property that they have no movable essential singularities, regardless of initial conditions; (3) To develop solutions to these O.D.E.'s which correspond to general initial conditions. In this regard the recent work of Flaschka and Newell [38] may be of interest. (4) Study the connection between the Bäcklund transformations developed in the Russian literature (see the review [39]) and by Fokas [40] and their connection to I.S.T. and monodromy preserving deformations.

(g) Discrete I.S.T. and Numerical Schemes

It is significant that many of the concepts related to the inverse scattering theory apply to suitably discretized nonlinear evolution equations; for example the Toda lattice, and discrete nonlinear Schrödinger equation (see for example [41], [42]). It is of interest to ask whether one can solve partial difference equations (i.e. numerical schemes) by inverse scattering. An obvious application would be to numerical simulations. We have succeeded in analytically developing such schemes [43]. These schemes can be shown to converge to a given nonlinear P.D.E. (which itself is solvable by inverse scattering) in the continuous limit. Moreover they have the nice property that they are neutrally stable, have exact soliton solutions and possess an infinite number of conserved quantities.

Recently we have (a student, T. Taha and myself) begun to compare the practical numerical simulation of a given nonlinear P.D.E. (e.g. cubic nonlinear Schrödinger or KdV) using traditional methods, with our newly developed schemes. In this sense, we hope to assess the usefulness of various numerical schemes on important model nonlinear problems. Preliminary results indicate very encouraging possibilities for these "new" schemes developed via I.S.T.

(h) Asymptotic Solutions

Aspects of the asymptotic solution of equations solvable by I.S.T. have been discussed by many authors. For a review of much of this work the reader may wish to see [37]. Despite all of the work already done on this question, the problem of long time asymptotic state evolving from initial data containing both solitons and the dispersive wavetrain remained unresolved. Indeed the separate questions of finding the asymptotic states evolving from initial conditions containing only pure solitons or pure dispersive waves had been solved. The difficulty that must be overcome when both states are present is that the solitons and dispersive wavetrain are of differing exponentially small asymptotic orders in certain regions of space. Recently [44] we were able to completely resolve this question for KdV. Moreover we find explicit formulae giving the phase shift of a soliton when it interacts with both solitons as well as the dispersive waves. A corollary to this result is the definition of a "perfect soliton' of an evolution equation; i.e. one which in the long time limit interacts elastically with <u>any</u> sufficiently localized disturbance.

We feel the ideas and methods described in [44] should apply in principle, to other nonlinear evolution equations solvable by I.S.T. We hope to investigate this question for some of the other nonlinear evolution equations solvable by I.S.T. (e.g. Nonlinear Schrödinger, Modified KdV, etc.)

(i) Recent Work by A. NAKAMURA

(a) A direct method of obtaining multiple periodic wave solutions

Recently the study of multiply periodic wave solutions to nonlinear evolution equations has been attracting the interest of numerous scientists. Here multiply periodic wave solutions correspond to the nonlinearly superposed state of several nonlinear periodic waves. This also corresponds to the periodic generalization of the so-called multiple soliton solutions of nonlinear evolution equations.

So far the analytic theory to calculate these multiply periodic wave solutions has been based on the theory of rather abstract multidimensional Riemann surfaces. Recently we have developed a completely independent and different approach to this problem, based on Hirota's direct method developed in soliton theory [45]. Just as the name indicates, our method is very direct. The direct method is constructed by elementary techniques. By this direct method, we have obtained multiply periodic wave solutions [46] of the intermediate long wave equation [47] which includes the KdV equation and Benjamin-Ono equation as limits. This intermediate long wave equation is considered to be important in the description of internal wave soliton pnenomena and has recently attracted wide interest [48].

At present, our direct approach however has been developed to calculate only up to a 2-periodic wave solution. We hope to extend our direct method to treat arbitrary N-periodic wave solutions.

(b) Cylindrical solitons, solitons in multi-dimensional systems

Investigations of solitons in multi-dimensional systems are important especially in view of the actual application to the real physical phenomena. Here just as the KdV equation has crucial importance as a model equation in one-dimensional nonlinear dispersive systems, the so-called cylindrical KdV [49] and spherical KdV [50] equations are important model equations respectively in two and three-dimensional nonlinear dispersive systems. Both cylindrical and spherical solitons have been studied experimentally and numerically [49,51]. However so far exact analytical solutions to these equations are either not known (in the case of spherical KdV equations) or could not reproduce experimental and numerical results (in the case of cylindrical KdV equation [53]). By generalizing Hirota's direct method appropriately, we have obtained a new analytic solution of the cylindrical KdV equation which does correctly reproduce experimental and numerical results [54]. We have also calculated the Bäcklund transformation for the cylindrical KdV equation [55]. These soliton solutions have a certain self similar character.

Although the cylindrical KdV equation has been successfully analyzed, the analogous solutions of the spherical KdV equation is not obtained by a straightforward application of the method to solve the cylindrical KdV equation. However since the cylindrical and spherical KdV equations are very similar to each other in form, there is a possibility that a generalization of the method to solve the cylindrical KdV equation can solve the spherical one. We are considering such a generalization.

c) Multiple decay mode solutions of nonlinear evolution equations.

In the case of the cylindrical KdV equation which has been discussed above, due to conservation of energy, the inwardly running "ring-form" soliton grows in amplitude and the outwardly running one decays. Thus it is natural to suspect that the similarity type solitons which appear in the cylindrical KdV equation have some relationship to decaying modes of other nonlinear evolution equations. In fact, guided by this consideration, very recently we have found certain similarity type multiple decay mode solutions of the two-dimensional KdV (Kadomtsev-Petviashvili) equation [56]. Furthermore it has been found that these newly found decaying mode solutions and usual soliton solutions can be nonlinearly superposed with each other. Further investigation of these and other solutions of two-dimensional KdV, and their relationship to I.S.T. is under consideration.

d) Chain of Bäcklund transformations

It is well known that the typical nonlinear evolution equations of current interest, KdV and modified KdV (=mKdV) equations are related with each other by the Miura transform or (non-auto) Bäcklund transform. In fact this process continues to other equations sequentially. To grasp general characteristics of this important Bäcklund transform, it is worthy to study this series further. We have recently carried out the study of such Bäcklund transform chains e.g. KdV→mKdV→"second mKdV"

[51]→"third mKdV"...[58] where arrows denote Bäcklund transforms. It is further hoped that from these studies, it will be possible to extract some meaningful information about the nature of the Bäcklund transform itself [58].

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1. Transverse Instability of One-Dimensional Transparent Optical Pulses in Resonant Media, Phys. Lett. 70A, #2, p. 83, 1979.

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29. Explode-decay mode lump solitons of a two-dimensional nonlinear Schrödinger equation, submitted to Phys. Letters, 1981.

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5. Invited Lectures, Conferences

Panelist on National Science Foundation Postdoctoral Fellowship Committee, New York City, February, 1979.

Mathematics Department, SUNY Buffalo, Buffalo, NY, April, 1979.

Conference on Inverse Scattering, Catholic University, Washington, DC, May, 1979.

Conference on Nonlinear Partial Differential Equations, University of Rhode Island, Kingston, Rhode Island, June, 1979.

Conference on Solitons and Related Phenomena, Jadwisin, Poland, August, 1979.

Joint U.S. - U.S.S.R. conference on Solitons, sponsored by the U.S. National Academy of Science and its counterpart in the U.S.S.R., Kiev, September, 1979.

Conference on Ill posed problems, University of Delaware, Newark, Delaware, October, 1979.

New York University, Courant Institute of Mathematical Sciences, December, 1979.

Columbia University, Department of Mathematics, February, 1979.

Workshop on Nonlinear Evolution Equations and Dynamical Systems, Chania, Crete, July 9-23, 1980.

Remarks on Nonlinear Evolution Equations and the Inverse Scattering Transform, Banff Conference, Banff Alberta, Canada, August, 1980.

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Panelist on National Science Foundation Postdoctoral Fellowship Committee, New York City, December, 1980.

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New York University, Courant Institute of Mathematical Science, 2 day conference in honor of K. O. Friedrichs, October, 1981.

form way, starting from an arbitrary classical Lie group, analytically solvable lattice models, among which is a far-reaching generalization of the XXX model, given by a quantum Hamiltonian H_{N-2}". The proposed S-matrix method will be instrumental in finding energy levels, and in the computation of correlation functions for the known and new lattice models of Toda and XXX type considered here.

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Linearization of the Korteweg–de Vries and Painlevé II Equations

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A new integral equation which linearizes the Korteweg-de Vries and Painlevé II equations, and is related to the potentials of the Schrödinger eigenvalue problem, is presented. This equation allows one to capture a far larger class of solutions than the Gel'fand-Levitan equation, which may be recovered as a special case. As an application this equation, with the aid of the classical theory of singular integral equations, yields a three-parameter family of solutions to the self-similar reduction of Korteweg -de Vries which is related to Painlevé II.

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Since the work of Gardner *et al.* in 1967,[1] there has been wide interest in the analysis of nonlinear evolution equations solvable by the so-called inverse-scattering transform (IST). The prototype example is the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

In this note we shall present a new linear integral equation which, in principle, allows one to capture a far larger class of solutions than does the Gel'fand-Levitan equation. Specifically we claim that if $\varphi(k;x,t)$ solves

$$\varphi(k;x,t) + i\exp[i(kx+k^3t)]\int_L \frac{\varphi(l;x,t)}{l+k}\,d\lambda(l) = \exp[i(kx+k^3t)], \tag{2}$$

where $d\lambda(k)$ and L are an appropriate measure and contour, respectively, then

$$u = -\frac{\partial}{\partial x}\int_L \varphi(k;x,t)\,d\lambda(k) \tag{3}$$

solves the KdV equation. The well-documented physical significance of the KdV equation, of its self-similar analogue, and of the associated Schrödinger scattering problem require us to attempt to characterize the form of the most general solution/potential possible.

We now enumerate the basic results given in this note. (i) We give a direct proof that (2) and (3) solve (1); (ii) we show how the well known Gel'fand-Levitan equation can be obtained from (2) as a special case; and (iii) we characterize by a matrix Fredholm equation a three-parameter family of solutions to the similarity ordinary differential equation of (1) which is directly related to the classical second equation of Painlevé (P II). We end with some remarks regarding the role of Bäcklund transformations and relevant generalizations.

We now consider (i). The point of view we take here is, in spirit, similar to that of Zakharov and Shabat.[2] Specifically, by direct calculation we show that solutions of (2) substituted in (3) satisfy (1). We make two assumptions: (a) $d\lambda$ and L are such that differentiation by x, t may be interchanged with $\int_L$; (b) the homogeneous integral equation has only the zero solution. Defining $\tilde{L}=\tilde{L}_0+3u\partial_x$, where $\tilde{L}_0=\partial_t+\partial_x^3$, after some manipulation we find

$$\tilde{L}\varphi(k;x,t)+i\exp[i(kx+k^3t)]\int_L\frac{\tilde{L}\varphi(l;x,t)}{l+k}d\lambda(l)=3k[k\varphi_x+i\varphi_{xx}+iu\varphi]. \qquad (4)$$

Similar calculations show that the quantity in brackets in the right-hand side of (4) satisfies the homogeneous integral equation. Hence $\tilde{M}\varphi=k\varphi_x+i\varphi_{xx}+iu\varphi=0$ which implies $\tilde{L}\varphi=0$, whereupon $\partial_x\int_L(\tilde{L}\varphi)d\lambda=0$ is (1). Moreover the equation $\tilde{M}\varphi=0$ is directly related to the Schrödinger eigenvalue problem. If we define

$$\varphi(k;x,t)=\psi(k;x,t)\exp[i(kx+k^3t)/2],$$

then $\tilde{M}\varphi=0$ gives

$$\psi_{xx}+(\tfrac{1}{2}k)^2\psi+u\psi=0. \qquad (5)$$

Next we pass on to (ii). The classical theory of inverse scattering and appropriately decaying solutions of KdV may be most easily obtained as follows. Let the measure $d\lambda(k)=r_0(\tfrac{1}{2}k)dk/2\pi$, where $r_0(k)$ is the usual reflection coefficient of $u(x,0)$ and the contour L goes over all the poles of $r_0(k)$. [Here we have assumed, for convenience, that $u(x,0)\to 0$ rapidly as $|x|\to\infty$.] Then substituting the expression for φ into (2), defining

$$K(x,y,t)=-(\tfrac{1}{2})\int_L\psi(k;x,t)\exp[i(ky+k^3t)/2]d\lambda(k),$$

and using

$$\exp[i(k+l)x/2]/(l+k)=-i\{\int_x^\infty\exp[i(k+l)\xi/2]/2d\xi\}$$

(k, l satisfy $\mathrm{Im}k, \mathrm{Im}l>0$), we obtain

$$K(x,y;t)+F(x+y;t)+\int_x^\infty K(x,\xi;t)F(\xi+y;t)d\xi=0, \qquad (6)$$

where

$$F(x,t)=(\tfrac{1}{2})\int_L\exp[i(kx/2+k^3t)]d\lambda(k),$$

and $u(x,t)=2\partial_x K(x,x;t)$. Hence by choosing the above measure $d\lambda$ and contour L, the Gel'fand-Levitan equation (6) may now be completely bypassed.

Soliton solutions of (1) may be calculated in a particularly easy manner from (2). Locations of the poles on the imaginary k axis in $r(k,0)$ correspond to soliton amplitudes, and the residues of $r(k,0)$ at these locations play the role of the normalization coefficients. Pure solitons may also be obtained by taking the measure as

$$d\lambda(k)_s=\sum_{j=1}^{N}c_j\delta(k-i\kappa_j)dk$$

(L passes through the $k=i\kappa_j$). Then (2) reduces to a linear *algebraic* system from which the well known N-soliton solution is immediately obtained.

We now discuss (iii). The KdV equation admits the similarity transformation $u(x,t)=U(x')/(3t)^{2/3}$, where $x'=x/(3t)^{1/3}$. The equation for U is given by (dropping the primes)

$$K_1(U)=U'''+6UU'-(2U+xU')=0. \qquad (7)$$

We note that (7) is directly related to P II:

$$P_2(V)=V''-xV-2V^3=\alpha. \qquad (8)$$

Specifically we note that the transformations $U=-V^2-V'$, $V=(U'+\alpha)/(2U-x)$ relate (8) to the equation

$$K_2(U) = U'' + 2U^3 - xU + [\nu + U' - (U')^2]/(2U - x) = 0$$

with $\nu = \alpha(\alpha+1)$. However, by direct calculation $[(2U-x)K_2(U)]' = (2U-x)K_1(U)$, hence $K_2(U)$ is an integral of (7), and thus there is a direct transformation between (7) and (8).[3] One may make use of these transformations fo find all the known (see, for example, Lukashevich[4] and Erugin[4]) elementary solutions of P II. Ablowitz and Segur[5] had established a connection between P II and IST and had characterized a one-parameter family of solutions via the Gel'fand-Levitan equation. Recently Flaschka and Newell[6] considered P II via monodromy theory. In the latter work the authors derived a formal system of linear singular integral equations for the general solution of P II. However, the highly nontrivial question of existence of solutions was left open.

An application of the result presented above in (i) is that a three-parameter family of solutions of (7) may be obtained from the linear singular integral equation

$$\varphi(t) + \frac{b(t)}{i\pi}\int_L \frac{\varphi(\tau)}{\tau+t}\,d\tau = f(t), \quad t \text{ on } L, \tag{9}$$

where $b(t) = f(t) = \exp[i(tx + t^3/3)]$ and $\int_L = \sum_{j=1}^{5} \hat{\rho}_j \int_{L_j}$ (see Fig. 1), $\hat{\rho}_1 = \hat{\rho}_2 = \rho_1$, $-\hat{\rho}_3 = \hat{\rho}_4 = \rho_2$, $\hat{\rho}_5 = \rho_3$. (Hereafter j always stands for $j = 1, \ldots, 5$). The solution to (7) is then obtained from

$$U = \frac{1}{\pi}\frac{\partial}{\partial x}\int_L \varphi(\tau)\,d\tau$$

(φ depends parametrically on x). We note that both (9) and U are obtained from (2) and (3) by a

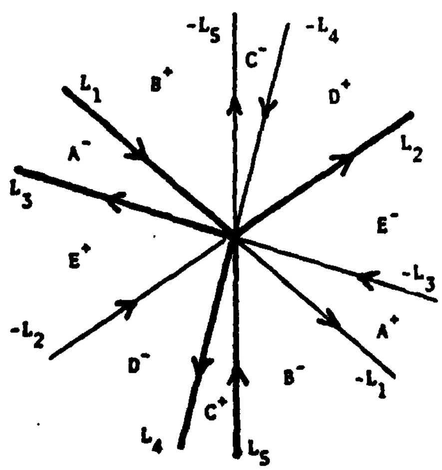

FIG. 1. Contours associated with Eqs. (9) and (12).

self-similar reduction. Moreover the contours L_j are obtained by finding the solution to the linear problem ($U = w_x$) $w''' - (w + xw') = 0$ in terms of integral representations and then deforming these contours so that they all pass through the origin. For example, note that $L_1 + L_2$ may be deformed to the usual Airy-function contour. If we restrict ourselves to this Airy contour, the result in Ref. 5 is obtained in the same manner as that in (ii) above.

We shall proceed to demonstrate that (9) may be reduced to a system of Riemann-Hilbert problems which are solvable using Fredholm theory. For this we need the full power of the classical theory of singular integral equations.[7-9]

Consider the sectionally holomorphic function

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\,d\tau. \tag{10}$$

The lines of discontinuity of $\Phi(z)$ are L_j; thus using the Plemelj formulas, we have

$$\Phi^+(t) = \Phi^-(t) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau, \quad t \text{ on } -L_j,$$
$$\Phi^\pm(t) = \pm\tfrac{1}{2}\hat{\rho}_j\varphi(t) + \frac{1}{2i\pi}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau, \quad t \text{ on } L_j, \tag{11}$$

where $\Phi^\pm(t)$ for t on L_j has the standard definitions[7-9] of limits of $\Phi(z)$ as $z \to t$ from the "left-hand side" (+) and "right-hand side" (−) of L_j, and where principal-value integrals are implied when needed. With use of (11), and Eq. (9) for t on L_j and $-t$ on $-L_j$, we obtain a system which we choose to write in the form

$$\underline{\Phi}^+(t) = G(t)\underline{\Phi}^-(t) + \underline{F}(t), \quad t \text{ on } \mathcal{L}_j, \tag{12}$$

where $\mathcal{L}_j = L_j + (-L_j)$, $\underline{\Phi}^+(t) = [\Phi^+(t), \Phi^-(-t)]^T$, $\underline{\Phi}^-(t) = \underline{\Phi}^+(-t)$, $\underline{F}(t) = [\bar{f}(t)H(t), -f(-t)H(-t)]^T$, $H(t) = \{\hat{\rho}_j$ if t on L_j, 0 if t on $-L_j\}$ and the components of the 2×2 matrix $G(t)$ are $G_{11}(t) = -2b(t) \times H(t) = -G_{22}(-t)$, $G_{12} = G_{21} = 1$.

One can prove the following statements.

(a) $\Phi^+(-t)$, $\Phi^-(-t)$ are "minus" and "plus" functions, respectively. (b) Necessary conditions for solvability of (12) are the symmetry conditions $G(t) = [G(-t)]^{-1}$, $\underline{F}(t) + G(t)\underline{F}(-t) = 0$, which are satisfied by the above $G, \underline{F}$. (c) Thus (12) defines a system of discontinuous Riemann-Hilbert problems with the additional restriction that $\underline{\Phi}^-(t) = \underline{\Phi}^+(-t)$. However this condition can be relaxed since one can show that (12) always admits a solution with this restriction, and moreover, in our case the solution is unique.

In order to solve (12) we first consider the homogeneous problem. The standard procedure is to transform the discontinuous homogeneous problem to a continuous one, and then obtain the fundamental set of solutions.

Associated with a given contour L_j, define the following auxilliary functions

$$\omega_{jk}^+(t) = \left(\frac{t}{t+z_j}\right)^{\lambda_{jk}}, \quad \omega_{jk}^-(t) = \left(\frac{t}{t-z_j}\right)^{\lambda_{jk}},$$
$$\omega_{jk} = \frac{\omega_{jk}^+}{\omega_{jk}^-}, \quad k=1,2, \tag{13}$$

where z_j is some j-dependent fixed point off L_j. The branches of the above functions are chosen such that ω_{jk}^+ and ω_{jk}^- are plus and minus functions, respectively (e.g., the branch cut for ω_{jk}^+ is taken between 0, $-z_j$ and hence lies to the right of L_j). The properties $\omega_{jk}(0+) = \exp[-i\pi\lambda_{jk}]$, $\omega_{jk}(0-) = \exp[i\pi\lambda_{jk}]$, $\omega_{jk}^+(t) = \omega_{jk}^-(-t)$ allow us to map the homogeneous system $\underline{\Phi}^+(t) = G(t)\underline{\Phi}^-(t)$ which has a discontinuity at $t=0$ to the following Riemann-Hilbert system which is continuous at the origin:

$$\underline{\Psi}^+(t) = g(t)\underline{\Psi}^-(t), \quad t \text{ on } L_j, \tag{14}$$

where we have used the transformation $\underline{\Phi}^+(t) = A\Omega^+(t)\underline{\Psi}^+(t)$, $\underline{\Phi}^-(t) = A\Omega^-(t)\underline{\Psi}^-(t)$ and hence $g(t) = [\Omega^+(t)]^{-1}A^{-1}G(t)A\Omega^-(t)$, with A, $\Omega^\pm(t)$ defined by

$$A_j = \begin{pmatrix} \frac{1-\Lambda_{j2}}{2\hat{\rho}_j}\exp[i\pi\lambda_{j1}/2] & \frac{1-\Lambda_{j1}}{2\hat{\rho}_j}\exp[i\pi\lambda_{j2}/2] \\ \exp[i\pi\lambda_{j1}/2] & \exp[i\pi\lambda_{j2}/2] \end{pmatrix}, \quad \Omega_j^\pm = \begin{pmatrix} \omega_{j\alpha}^\pm & 0 \\ 0 & \omega_{j\beta}^\pm \end{pmatrix}, \tag{15}$$

where for $j=2,3,4$ we have $\alpha=1$, $\beta=2$ and for $j=1,5$ we have $\alpha=2$, $\beta=1$; the λ_{jk} and Λ_{jk} are defined by

$$\exp[i\pi\lambda_{j1}] = \hat{\rho}_j + (1+\hat{\rho}_j^2)^{1/2}, \quad \exp[i\pi\lambda_{j2}] = -\hat{\rho}_j + (1+\hat{\rho}_j^2)^{1/2}, \quad \Lambda_{j1} = \exp[2i\pi\lambda_{j1}], \quad \Lambda_{j2} = \exp[2i\pi\lambda_{j2}].$$

The matrix $g(t)$ has the properties $g(t) = [g(-t)]^{-1}$ and $\det g = -1$.

One may characterize a solution of the system (14) by imposing the condition $\underline{\Psi}(z) - \underline{\Psi}_A^- = \underline{\gamma}$ as $|z| \to \infty$ in A^-. This leads to a Fredholm equation for, say, $\underline{\Psi}^+(t)$, which however must be interpreted in a suitable principal-value sense as it does not converge in the normal sense at infinity. Alternatively, one may obtain a regular Fredholm equation of the second kind by imposing conditions at a finite point off all contours, say $z=1$. This leads to the following Fredholm equation for $\underline{\Psi}^+(t)$:

$$\underline{\Psi}^+(t) + \frac{1}{2\pi i}\int_L \left[\frac{1}{\tau - t} - \frac{1}{\tau - 1}\right][g_j(t)g(-\tau) - I]\underline{\Psi}^+(\tau)d\tau = g_j(t)\underline{\beta}, \quad t \text{ on } L_j, \tag{16}$$

where $\underline{\beta} = \underline{\Psi}(1)$, $\int_L = \sum_{j=1}^{3}(\int_{L_j} + \int_{-L_j})$, and I is the unit matrix. Any two linearly independent $\underline{\beta}$ vectors, say $\underline{\beta}_{1,2}$, lead to a fundamental matrix $Y^+(t) = [\underline{\Psi}_1^+(t), \underline{\Psi}_2^+(t)]$ for the system (14).

With use of the above results the fundamental matrix of the discontinuous problem (12) is given by

$$X^+(t) = A\Omega^+(t)[\underline{\Psi}_1^+(t), \underline{\Psi}_2^+(t)]. \tag{17}$$

Hence the solution of (12) is given by

$$\underline{\Phi}^+(t) = \frac{\underline{F}(t)}{2} + \frac{1}{2\pi i}X^+(t)\int_L \frac{[X^+(\tau)]^{-1}\underline{F}(\tau)}{\tau - t}d\tau. \tag{18}$$

Having obtained $\underline{\Phi}^+(t)$ and using (11) to obtain $\varphi(t)$, we have characterized a three-parameter family of solutions of U. With use of the results of Fredholm's theory the nonmovable critical-point property of U is easily verified.

Finally, we make some remarks. First, we only expect from (2) to obtain solutions to P II in the range $-\frac{1}{2} < \alpha < \frac{1}{2}$. To obtain the solution for all ranges of α, we believe, the Bäcklund transformations (following Rosales[10]) and "finite perturbations" (see, for example, Ablowitz and Cornille[11]) of suitable elementary solutions must be employed. Similarly, wider classes of solutions to KdV should be obtainable this way (we shall remark on this more completely in the future). Second, straightforward generalizations to the higher-order KdV equations, as well as to many other nonlinear evolution equations, are possible. Third, motivation for some of the ideas in this note originate from the concept of summing perturbation series. Relevant perturbation series can be readily developed (see, for example, Refs. 10 and 12).

END

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